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Distributional chaos and spectral decomposition on the circle ☆

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Abstract

Schweizer and Smítal [Trans. Amer. Math. Soc. 344 (1994) 737–754] introduced the notion of distributional chaos for continuous maps of the interval. In this paper we show that similar results, up to natural modifications, are valid for the continuous mappings of the circle. Thus any such map has a finite spectrum, which is generated by the map restricted to a finite collection of basic sets, and any scrambled set in the sense of Li and Yorke has a decomposition into three subsets (on the interval into two subsets) such that the distribution function generated on any such subset is bounded from below by a distribution function from the spectrum. While the results are similar, the original argument is not applicable directly and needs essential modifications.

MSC: 37E10

Keywords: Dynamical system; Maximal ω-limit set; Basic set; Circle

1. Introduction

Let (X, ϱ) be a compact metric space. For f in the space $C(X, X)$ of continuous mappings from X into itself, $x, y \in X$, real t, and any positive integer n define

$$\xi(x, y, t, n) = \sum_{i=0}^{n-1} \chi_{[0,t)}\bigl(\delta_{xy}(i)\bigr) = \#\bigl\{i\colon\ 0 \leqslant i < n \text{ and } \delta_{xy}(i) < t\bigr\}, \tag{1}$$

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$$F^*_{xy}(t) = \limsup_{n\to\infty} \frac{1}{n}\xi(x, y, t, n), \tag{2}$$

and

$$F_{xy}(t) = \liminf_{n\to\infty} \frac{1}{n}\xi(x, y, t, n), \tag{3}$$

where $\delta_{xy}(i) = \varrho(f^i(x), f^i(y))$, and χ_A is the characteristic function of the set A.

Clearly both F^*_{xy}, F_{xy} are nondecreasing functions such that $F^*_{xy}(t) = F_{xy}(t) = 0$ for $t < 0$, and $F^*_{xy}(t) = F_{xy}(t) = 1$ for $t > \operatorname{diam} X$. We identify any two nondecreasing functions that coincide everywhere except at a countable set, and adopt the convention to choose functions F^*_{xy}, F_{xy} as left-continuous. Functions F^*_{xy}, F_{xy} are called the *upper* and *lower distribution function* of x and y, respectively. A function f *exhibits distributional chaos* if there are points $x, y \in \mathbb{S}$ such that $F^*_{xy}(t) = 1$ for all $t > 0$ and there is a point $s \in (0, 1)$ such that $F^*_{xy}(s) > F_{xy}(s)$.

In the special case $X = I$ distributional chaos has many nice, "regular" properties [18] which disappear on other compact metric spaces. See, e.g., the survey paper [19], or recent results [7] or [8]. In particular, distributional chaos on the interval appears if and only if the map has positive topological entropy [18]. This characterization is not true on more general spaces [13]. Even in the case $X = I^2$ there are distributionally chaotic triangular maps with zero topological entropy [9], or with other irregularities [1]. However, on the circle distributional chaos has similar "regularity" properties as on the interval.

Throughout the paper we use the standard terminology and notation. In particular, $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ is the circle, and $\pi : \mathbb{R} \to \mathbb{S}$ the natural projection defined by $\pi(x) = x - [x]$, where $[x]$ is the integer part of x. We denote by $C(\mathbb{S}, \mathbb{S})$ the set of continuous mappings of $\mathbb{S}$ into itself. The metric on $\mathbb{S}$ is given by $\varrho(x_1, x_2) = \min\{|y_1 - y_2|\colon\ x_i = \pi(y_i)\}$. For $X = \mathbb{S}$ distributional chaos appears if and only if $h(f) > 0$, or equivalently if f has a basic set. See [14].

We use the following terminology. For $f \in C(\mathbb{S}, \mathbb{S})$, let $\omega_f(x)$ denote the ω-limit set of x. Points $x, y \in \mathbb{S}$ are *synchronous* if the sets $\omega_f(x)$ and $\omega_f(y)$ are contained in the same maximal ω-limit set ω and if, for any periodic interval J such that its orbit $\operatorname{Orb} J$ contains ω, there is a $j \geqslant 0$ such that $f^j(x), f^j(y) \in J$. The *spectrum* $\Sigma(f)$ of f is the set of minimal elements of the set $D(f) = \{F_{xy}\colon\ x \text{ and } y \text{ are synchronous}\}$. And the *weak spectrum* $\Sigma_w(f)$ of f is the set of minimal elements of the set $D_w = \{F_{xy}\colon\ \liminf_{i\to\infty} \delta_{xy}(i) = 0\}$. We will see that, for a continuous map f of the circle, both the spectrum and the week spectrum are nonempty and finite (similarly as on the interval). A *scrambled set* (in the sense of Li and Yorke) is any set $S \subset \mathbb{S}$ such that, for any distinct points x and y in $\mathbb{S}$,

$$\liminf_{i\to\infty} \delta_{xy}(i) = 0, \qquad \limsup_{i\to\infty} \delta_{xy}(i) > 0.$$

Distributional chaos on the interval is supported by *basic sets*, i.e., maximal infinite ω-limit sets containing a periodic point. The properties of basic sets on the interval are wellknown. Many of them were discovered by Sharkovsky (cf., e.g., [15–17]). Detailed study of basic sets and solenoids (i.e., infinite maximal ω-limit sets containing no periodic points) on the interval, extending some of Sharkovsky's results can be found in [2].

As we show, basic sets are essential also for distributional chaos on the circle. We can adopt the same definition as on the interval [14]. Thus, a basic set is any infinite maximal ω-limit set which contains a periodic point. (For more details see Section 3.)

The paper is organized as follows. In the next section we present the main results concerning distributional chaos. Section 3 contains a classification of ω-limit sets. In Section 4 we recall some not widely known results of Blokh [3–6] concerning basic sets. Using this and some easily verifiable facts we obtain summarizing Theorem 4.10 which is interesting in itself and is applied in the remainder of the paper. Sections 5 and 6 contain auxiliary results on distributional chaos, and the last Section 7 the proofs of the main Theorems A and B.

2. Main results

To summarize the main results, we first describe the dynamics on a single basic set (Theorem A) and then the general case (Theorem B). Recall that part (A) of Theorem B was already proved in [14].

Theorem A. *Let $f \in C(\mathbb{S}, \mathbb{S})$ and let $\tilde{\omega}$ be a basic set of f. Then there are a nondecreasing function $F : \mathbb{R} \to [0, 1]$, a nonempty perfect set $P \subset \tilde{\omega}$, and a positive ε with the following properties*:

(i) $F(\varepsilon) = 0$ *and* $\Sigma(f|_{\tilde{\omega}}) = \{F\}$;
(ii) $F = F_{xy} < F^*_{xy} = \chi_{(0,\infty)}$ *for any* $x \neq y$ *in* P;
(iii) *if S is a scrambled set for f such that $\omega_f(x) \subset \tilde{\omega}$ for any $x \in S$ then there are sets S_0, S_1, S_2 ($S_0 \neq \emptyset$), such that $S = S_0 \cup S_1 \cup S_2$ and $F_{xy} \geqslant F$ whenever $x, y \in S_k$.*

Theorem B. *Let $f \in C(\mathbb{S}, \mathbb{S})$.*

(A) *If the topological entropy of f is zero, then $\Sigma(f) = \Sigma_w(f) = \{\chi_{(0,\infty)}\}$.*
(B) *If the topological entropy of f is positive, then*:
  (B1) *Both the spectrum $\Sigma(f)$ and the weak spectrum $\Sigma_w(f)$ are finite and nonempty. Specifically $\Sigma(f) = \{F_1, \dots, F_m\}$ for some $m \geqslant 1$, and $\Sigma \setminus \Sigma_w(f) = \{F_{m+1}, \dots, F_n\}$ where $n \geqslant m$. Furthermore, for each i there is an $\varepsilon_i > 0$ such that $F_i(\varepsilon_i) = 0$.*
  For any positive integer $k \leqslant n$, let π_k be the system of sets P such that $\#P \geqslant 2$ and for any distinct u, v in P, $F_k = F_{uv} < F^*_{uv} = \chi_{(0,\infty)}$.
  (B2) *If $k \leqslant m$, then π_k contains a nonempty perfect set P_k.*
  (B3) *If, on the other hand, $m < k \leqslant n$ then π_k is nonempty and any P in π_k contains two or three points.*
  (B4) *If S is a scrambled set for f (or more generally if, for any u, v in S, $\liminf_{i\to\infty} \delta_{uv}(i) = 0$), then there are integers $i, j, k \leqslant m$ and a decomposition $S = S_i \cup S_j \cup S_k$ such that $F_{uv} \geqslant F_l$ if $u, v \in S_l$, for $l \in \{i, j, k\}$.*

3. Classification of maximal ω-limit sets

By a maximal ω-limit set we mean any ω-limit set which in not properly contained in an another ω-limit set. The following result is a consequence of Theorem 4 in [6].

Theorem 3.1. *Any ω-limit set of $f \in C(\mathbb{S},\mathbb{S})$ is contained in a maximal ω-limit set.*

Let $f \in C(\mathbb{S},\mathbb{S})$, and let U be an open interval. Put

$$P_U = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} f^k(U). \tag{4}$$

If $\omega = \omega_f(x)$ is an ω-limit set, and U is an interval whose interior intersects ω then $f^k(U) \cap U \neq \emptyset$ for some $k \in \mathbb{N}$ and hence, P_U has finite number $n(U)$ of connected components. If $\lim_{|U|\to 0} n(U) = \infty$ then ω is infinite; it is called *solenoid*. Such a set contains no periodic points. Otherwise, ω has a decomposition into $n = \lim_{|U|\to 0} n(U)$ *periodic portions* forming a unique orbit of period n. Clearly, the same is true for the maximal ω-limit set $\tilde{\omega}$ containing ω (cf. Theorem 3.1).

If $\tilde{\omega}$ is finite it is a periodic orbit. If $\tilde{\omega}$ is infinite and contains a periodic point then it is called *basic set*. Finally, if $\tilde{\omega}$ is infinite and contains no periodic point then we call it a *singular set*. Recall that the last case is impossible on the interval while, on the circle, an irrational rotation generates such ω-limit set. Conversely, f restricted to a singular set is semiconjugate to an irrational rotation: if f has a singular set, then it has no periodic points [14], and if a map in $C(\mathbb{S},\mathbb{S})$ has no periodic points then it is semiconjugate to an irrational rotation [12].

4. Properties of basic sets

The main result in this section is Theorem 4.10. It is a consequence of results by Blokh [3–6] which are not widely known. We explicitly state Theorems 4.3 and 4.9, similarly as Theorem 3.1 in the previous section. Other results stated below are contained in the quoted Blokh's results implicitly; for completeness we present them with simple arguments.

We start with the next lemma. Recall that P_U is given by (4).

Lemma 4.1. *Let $f \in C(\mathbb{S},\mathbb{S})$, and let U be a nonwandering interval. Then P_U is f-invariant and has finite number of connected components.*

Proof. Since U is nonwandering P_U has finite number of components, and $f(P_U) = f(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} f^k(U)) = \bigcap_{n=1}^{\infty} \bigcup_{k=n+1}^{\infty} f^k(U) = P_U$. □

Let ω be an ω-limit set for $f \in C(\mathbb{S},\mathbb{S})$. We say that interval U is *essential for ω* (or *ω-essential*) if its interior intersects ω. Note that any ω-essential interval is nonwandering.

By Lemma 4.1, P_U consists of finite number of disjoint periodic intervals. These intervals form a unique orbit, and their closures intersect ω in compact *periodic portions*. The compact periodic portions of ω, determined by a given interval U, form a *periodic decomposition* of ω.

For a basic set $\tilde{\omega}$ define the *envelope* $\mathrm{Env}(\tilde{\omega})$ of $\tilde{\omega}$ as follows. If $\overline{P_U} = \mathbb{S}$ for every $\tilde{\omega}$-essential interval U, put $\mathrm{Env}(\tilde{\omega}) = \mathbb{S}$. Otherwise, let U be a sufficiently small interval such that $n(U) = \max\{n(V)\colon \mathrm{int}(V) \cap \tilde{\omega} \neq \emptyset\} < +\infty$. By Lemma 4.1, $P_U = J_1 \cup \cdots \cup J_{n(U)}$. Put $\mathrm{Env}(\tilde{\omega}) = H_1 \cup \cdots \cup H_{n(U)}$ where $H_i \subset \overline{J_i}$ is the minimal compact interval such that $\overline{J_i} \setminus H_i$ is disjoint from $\tilde{\omega}$. We say, that f is an *$\tilde{\omega}$-finitely covering map* if for every $\tilde{\omega}$-essential interval U there is an $n \in \mathbb{N}$ such that $\bigcup_{j=0}^{n} f^j(U) = \mathbb{S}$. Thus, if f is finitely covering then $\mathrm{Env}(\tilde{\omega}) = \mathbb{S}$.

Lemma 4.2. *Let $f \in C(\mathbb{S}, \mathbb{S})$ and $u, v \in \mathbb{S}$. Let $\{U_i\}_{i=0}^{\infty}$, $\{V_i\}_{i=0}^{\infty}$ be compact intervals with $\lim_{i\to\infty} U_i = U$ and $\lim_{i\to\infty} V_i = V$, where U, V are sets (obviously compact). Let for any i and j there exist positive integers $u(i,j)$ and $v(i,j)$ such that $f^{u(i,j)}(U_i) \supset V_j$ and $f^{v(i,j)}(V_i) \supset U_j$. Then there are $u \in U$ and $v \in V$ such that $\{u, v\} \subset \omega_f(y)$, for some $y \in \mathbb{S}$.*

Proof. Define a decreasing sequence $\{J_i\}_{i=1}^{\infty}$ of compact intervals and an increasing sequence $\{n(i)\}_{i=1}^{\infty}$ of positive integers as follows: $J_1 = U_1$ and $n(1) = u(1,2)$. Then $f^{n(1)}(J_1) \supset V_2$; choose $J_2 \subset J_1$ such that $f^{n(1)}(J_2) = V_2$. Take $n(2) = n(1) + v(2,3)$. Then $f^{v(2,3)}(V_2) \supset U_3$ and there is a $J_3 \subset J_2$ such that $f^{n(2)}(J_3) = U_3$. Then there are J_4 and $n(3)$ such that $f^{n(3)}(J_4) = V_4$, etc. Let $y \in \bigcap_{i=1}^{\infty} J_i$. Since $\mathbb{S}$ is compact and the trajectory of y visits every neighborhood of U and every neighborhood of V the result follows. $\square$

Theorem 4.3. *Any basic set $\tilde{\omega}$ of an $f \in C(\mathbb{S}, \mathbb{S})$ is perfect.*

Proof. Follows from Theorem 2 in [4]. $\square$

The next statements and/or their proofs deal with indecomposable basic sets $\tilde{\omega}$, i.e., basic sets $\tilde{\omega}$ such that have periodic decomposition into one portion (P_U is an interval or the circle for any open interval intersecting $\tilde{\omega}$).

Lemma 4.4. *Let $f \in C(\mathbb{S}, \mathbb{S})$ and $\tilde{\omega}$ be an indecomposable basic set for f. Then there is an invariant interval $K \subset \mathbb{S}$ (or the circle) covering $\tilde{\omega}$ and $g : K \to K$ continuous transitive such that f is semiconjugate to g via $\varphi \in C(\mathbb{S}, \mathbb{S})$. The map φ is one-to-one on $\mathbb{S}$, except for the closures of the intervals contiguous to $\tilde{\omega}$, which are the intervals of constancy of φ. Moreover if f is not an $\tilde{\omega}$-finitely covering map then K can be chosen such that $\overline{K} \neq \mathbb{S}$.*

Proof. Theorem 2 in [4] gives us K and g. To prove the second statement note that if f is not $\tilde{\omega}$-finitely covering then for any open $\tilde{\omega}$ essential interval U, $P_U \neq \mathbb{S}$. Thus $\overline{K} \subset P_U \neq \mathbb{S}$. $\square$

Theorem 4.5. *If $\tilde{\omega}$ is a basic set for $f \in C(\mathbb{S},\mathbb{S})$and $\omega_f(x) \subset \tilde{\omega}$, then the set $A = \{v \in \tilde{\omega}\colon \omega_f(v) = \omega_f(x)\}$ is dense in $\tilde{\omega}$.*

Proof. We may assume that $\tilde{\omega}$ is indecomposable (otherwise, if $\tilde{\omega}$ is decomposable into n portions we may replace f by f^n and get that A is dense in each portion of $\tilde{\omega}$).

Using Lemma 4.4 we get that f is semiconjugate to continuous transitive $g : K \to K$. If K is an interval ($K \neq \mathbb{S}$) then we can reduce our problem to the map of the interval.

If $K = \mathbb{S}$ then $\varphi(\tilde{\omega}) = \mathbb{S}$ and $\varphi(\omega_f(x)) = \omega_g(\varphi(x))$. Take $J \subset \mathbb{S}$ arbitrary open interval intersecting $\tilde{\omega}$ and set $H = \varphi(J)$. By Theorem 4.3 and Lemma 4.4, H is a (nondegenerate) interval. Since g is transitive, there is a point $u \in H$ and a $k \in \mathbb{N}$ such that $g^k(u) = \varphi(x)$. Take $v \in \varphi^{-1}(u) \cap \tilde{\omega}$. Then $v \in J$ and $\omega_f(v) = \omega_f(x)$. □

Theorem 4.6. *Let $f \in C(\mathbb{S},\mathbb{S})$ and let $\tilde{\omega}$ be a basic set for f. If J is an interval such that $\tilde{\omega} \cap J$ is infinite then $\tilde{\omega} \cap J$ contains a periodic point.*

Proof. It is sufficient to prove theorem for the case where $\tilde{\omega}$ is indecomposable (otherwise we can take f^n instead of f and appropriate portion of $\tilde{\omega}$ instead of $\tilde{\omega}$).

Recall that theorem is true for basic sets on the interval, cf. [18]. Using Lemma 4.4 we get that f is semiconjugate to continuous transitive $g : K \to K$. If K is an interval ($K \neq \mathbb{S}$) then we can reduce our problem to the map of the interval.

If $K = \mathbb{S}$ then $g \in C(\mathbb{S},\mathbb{S})$ and a semiconjugating map $\varphi \in C(\mathbb{S},\mathbb{S})$ is such that $\varphi \circ f = g \circ \varphi$. Since $\tilde{\omega}$ contains a periodic point of f then $\mathrm{Per}(g) \neq \emptyset$. By Theorem 7.2 in [11], the set $\mathrm{Per}(g)$ is dense in $\mathbb{S}$. By Lemma 4.4, $\varphi(J)$ is an interval. Consequently, $\tilde{\omega} \cap J$ contains a periodic point of f. □

Lemma 4.7. *Let $f \in C(\mathbb{S},\mathbb{S})$ and let $\tilde{\omega}$ be an indecomposable basic set for f. Then for every compact interval $K \subset \mathrm{int}(\mathrm{Env}(\tilde{\omega}))$ and every compact interval J such that $J \cap \tilde{\omega}$ is infinite, there is $k \in \mathbb{N}$ such that $f^k(J) \supset K$.*

Proof. We may assume that $J \subset \mathrm{Env}(\tilde{\omega})$. By Theorem 4.6 and Theorem 4.3 there are periodic points $p, q \in J$ such that $[p,q] \subset J$ and $[p,q] \cap \tilde{\omega}$ is infinite. Let $m > 0$ be the common multiple of the periods of p and q. Then $f^m([p,q]) \supset [p,q]$, and $L = \bigcup_{i=1}^{\infty} f^{im}([p,q])$ is a periodic interval with $f^m(L) = L$. We then must have $\tilde{\omega} \subset \overline{L}$, since $\tilde{\omega}$ is indecomposable (cf. Lemma 3.2 in [18]). Thus $\mathrm{Env}(\tilde{\omega}) \subset \overline{L}$ and consequently, $f^k([p,q]) \supset K$ for any sufficiently large k. □

Lemma 4.8. *Let $f \in C(\mathbb{S},\mathbb{S})$, let $\tilde{\omega}_1$ and $\tilde{\omega}_2$ be two different indecomposable basic sets and $U = \mathrm{Env}(\tilde{\omega}_1)$, $V = \mathrm{Env}(\tilde{\omega}_2)$. Then one of the following conditions holds*:

(i) *$U \subset \mathrm{int}(V)$ or $V \subset \mathrm{int}(U)$;*
(ii) *$U \cap V = \emptyset$;*
(iii) *U, V have at most two points in common.*

In particular, $U \neq V$.

Proof. If U, V have more than two points in common then $U \cap V \supset J$, where J is an interval and if neither $U \subset V$ nor $V \subset U$, then $J \cap \tilde{\omega}_1$ and $J \cap \tilde{\omega}_2$ are uncountable (Theorem 4.3). Let $u \in \tilde{\omega}_1$ and $v \in \tilde{\omega}_2$ be points interior to J such that $\omega_f(u) = \tilde{\omega}_1$ and $\omega_f(v) = \tilde{\omega}_2$ (Theorem 4.5). By Lemmas 4.7 and 4.2 $u, v \in \omega_f(z)$, for some z, and since $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are maximal, we have $\tilde{\omega}_1 = \tilde{\omega}_2$—a contradiction. □

Theorem 4.9. *The system of all basic sets for $f \in C(\mathbb{S}, \mathbb{S})$ is countable.*

Proof. It follows from Theorem 4 in [6]. □

The following theorem summarizes properties of basic sets obtained in this section.

Theorem 4.10. *Let $f \in C(\mathbb{S}, \mathbb{S})$, $x \in \mathbb{S}$ and let $\tilde{\omega}$ be a basic set.*

(i) *$\tilde{\omega}$ is perfect*;
(ii) *if $\omega_f(x) \subset \tilde{\omega}$, then $\{y \in \tilde{\omega}\colon \omega_f(y) = \omega_f(x)\}$ is dense in $\tilde{\omega}$*;
(iii) *if J is an interval such that $J \cap \tilde{\omega}$ is infinite then $\tilde{\omega} \cap J$ contains a periodic point*;
(iv) *the system of basic sets of f is countable*;
(v) *if $\tilde{\omega}_1 \neq \tilde{\omega}_2$ are indecomposable basic sets and $U = \mathrm{Env}(\tilde{\omega}_1)$, $V = \mathrm{Env}(\tilde{\omega}_2)$, then $U \cap V = \emptyset$, or U and V have at most two points in common, or $U \subset \mathrm{int}(V)$, or $V \subset \mathrm{int}(U)$; in particular, $U \neq V$*;
(vi) *if $\tilde{\omega}$ is indecomposable then, for every compact interval K contained in the interior of $\mathrm{Env}(\tilde{\omega})$, and every compact interval J such that $J \cap \tilde{\omega}$ is infinite, there is a $k \in \mathbb{N}$ such that $f^k(J) \supset K$. We describe this situation saying that $f|_{\tilde{\omega}}$ is strongly transitive.*

Proof. (i) follows from Theorem 4.3, (ii) follows from Theorem 4.5, (iii) follows from Theorem 4.6, (iv) follows from Theorem 4.9, (v) follows from Theorem 4.8, and (vi) follows from Lemma 4.7. □

5. Distributional chaos on basic sets

Lemma 5.1. *Let $\tilde{\omega}$ be an indecomposable basic set for $f \in C(\mathbb{S}, \mathbb{S})$ and let $U = \mathrm{Env}(\tilde{\omega})$. Let $\{J_n\}_{n \in K}$, $J_n \subset U$, be an enumeration of intervals complementary to $\tilde{\omega}$. Then, for any $n \in K$, either $f(J_n)$ is a singleton or there is a $k \in K$ such that $f(J_n) \subset \overline{J_k}$.*

Proof. Lemma is a consequence on Lemma 4.4 □

Proof of the main result in [18] is based on the following result stated by Sharkovsky [15] in 1966. *If $\tilde{\omega}$ is a basic set for a map $f \in C(I, I)$, and $\omega_f(x) \subset \tilde{\omega}$ for some $x \in \mathbb{S}$, then there is a $k > 0$ with $f^k(x) \in \tilde{\omega}$.* Unfortunately, this result is wrong. To see this let $f \in C(I, I)$ be transitive on $[0, 1/2]$, with $f[0, 1/2] = [0, 1/2]$, $f(1/2) = 1/2$, and let $f(x) = x/2 + 1/4$ for $x \in [1/2, 1]$. Then $\tilde{\omega} = [0, 1/2]$ is a unique basic set for f, $\omega_f(1) = \{1/2\} \subset \tilde{\omega}$ but $f^k(1) \notin \tilde{\omega}$ for all $k > 0$. However, we can replace the wrong result by its modification which works also in the case $X = I$.

Lemma 5.2. *Let $\tilde{\omega}$ be a basic set for $f \in C(\mathbb{S}, \mathbb{S})$, and let $\omega_f(x) \subset \tilde{\omega}$. Then there is a $y \in \tilde{\omega}$ such that* $\lim_{i\to\infty} \delta_{xy}(i) = 0$ *and hence, $\omega_f(x) = \omega_f(y)$.*

Proof. Let $f^n(x) \notin \tilde{\omega}$ for all n. We may assume that $\omega_f(x)$ is infinite since otherwise it is a cycle, and that $\tilde{\omega}$ is nowhere dense since otherwise $\tilde{\omega}$ consists of a finite number of compact intervals. In both cases the result would follow immediately.

Since $\omega_f(x)$ is infinite $f^n(x)$ is in an interval $B \subset \mathrm{Env}(\tilde{\omega})$ complementary to $\tilde{\omega}$, for some $n \geqslant 0$; we may assume $x \in B$. By Lemma 5.1 the closure $\overline{B}$ of B is wandering or periodic. In the first case $\lim_{i\to\infty} \mathrm{diam}(f^i(B)) = 0$ and as y we take an endpoint of B. If $\overline{B}$ is periodic then $\omega_f(x)$ would be finite. □

Lemma 5.3. *Let $f \in C(\mathbb{S}, \mathbb{S})$ and let $\{\omega_i\}_{i=1}^{\infty}$ be a sequence of distinct minimal* (*i.e., indecomposable*) *periodic portions of basic sets of f. If the periods of ω_i are bounded then* $\lim_{i\to\infty} \mathrm{diam}\,\omega_i = 0$.

Proof. Let m be a common multiple of the periods of all ω_i and $K_i = \mathrm{Env}(\omega_i)$. Assume that the lemma is not true. Replacing f by f^m we can assume that $m = 1$ and that, for any i, $\mathrm{diam}\, K_i > \varepsilon$, where ε is positive. By (v) of Theorem 4.10 it suffices to consider the case when $K_1, K_2, \ldots$ is a monotone sequence. We may assume that it is a decreasing sequence, since in the other case the argument is similar.

Choose $\delta > 0$ such that $\mathrm{diam}\, f(A) < \varepsilon$ for any set A with $\mathrm{diam}\, A < \delta$. Again by (v) of Theorem 4.10 assume that $K_i = [a_i, b_i] \neq \mathbb{S}$ and $0 \notin K_i$, for every i. Then $a_i < a_{i+1}$ and $b_{i+1} < b_i$. Let $\omega^0 = [a_1, a_2] \cap \omega_1$ and $\omega^1 = [b_2, b_1] \cap \omega_1$. Since $\omega_1 = \omega^0 \cup \omega^1$ is indecomposable, one of the sets, $[a_1, a_2]$, $[b_1, b_2]$ say $[a_1, a_2]$ is mapped by f over $[a_2, b_2]$. Thus $|a_2 - a_1| > \delta$ and $\mathrm{diam}\, K_2 < \mathrm{diam}\, K_1 - \delta$. By induction we get $\mathrm{diam}\, K_{i+1} < \mathrm{diam}\, K_1 - i\delta$, for any i, which is impossible. □

The next three lemmas are slight modifications of results proved in [18] for the interval mappings. For the reader's convenience we insert the arguments which are very simple, and are almost the same as in the original paper.

Lemma 5.4. *Let $f \in C(\mathbb{S}, \mathbb{S})$. Then for any t, λ in $(0, 1)$ there is an integer $n(t, \lambda)$ with the following property*: *If A is a periodic set of period $m \geqslant n(t, \lambda)$, and convex hulls of sets $f^s(A)$ for $s < m$, are nonoverlapping, then for any u, v in A, $F_{uv}(t) > \lambda$.*

Proof. Fix t and λ. Let $n(t, \lambda)$ be such that $(n(t, \lambda) - 1/t)/n(t, \lambda) > \lambda$. Since there are at most $1/t$ distinct sets $f^s(A)$ with $\mathrm{diam}\, f^s(A) \geqslant t$ we have $F_{uv}(t) \geqslant 1/m \cdot \#\{s < m$: $\mathrm{diam}\, f^s(A) < t\} \geqslant (m - 1/t)/m > \lambda$. □

Lemma 5.5. *Let $f \in C(\mathbb{S}, \mathbb{S})$ and both F^*_{xy} and F_{xy} be continuous at $t \in (0, 1)$. Then, for any $\varepsilon > 0$, there are arbitrarily large positive integers k, q, and $\delta > 0$ such that*

$$\frac{1}{k}\xi(u, v, k, t) < F_{xy}(t) + \varepsilon$$

and

$$\frac{1}{q}\xi(u,v,q,t) > F^*_{xy}(t) - \varepsilon$$

whenever $\varrho(u,x) < \delta$ *and* $\varrho(v,y) < \delta$.

Proof. Choose $\varepsilon_1 > 0$ such that $F_{xy}(t+2\varepsilon_1) < F_{xy}(t) + \varepsilon/2$ and $F^*_{xy}(t-2\varepsilon_1) > F^*_{xy}(t) - \varepsilon/2$. Then choose $k \in \mathbb{N}$ such that $1/k \cdot \xi(x,y,k,t+2\varepsilon_1) < F_{xy}(t+2\varepsilon_1) + \varepsilon/2$. The first equality follows from the fact that $\xi(u,v,k,t) \leqslant \xi(x,y,k,t+2\varepsilon_1)$ whenever $\delta > 0$ is sufficiently small. The argument for the second inequality is similar. □

Lemma 5.6. *Let* $f \in C(\mathbb{S},\mathbb{S})$, *let* $\tilde{\omega}_1, \tilde{\omega}_2$ *be basic sets, and let* U *and* V *be the minimal compact periodic intervals with* $\mathrm{Orb}(U) \supset \tilde{\omega}_1$ *and* $\mathrm{Orb}(V) \supset \tilde{\omega}_2$. *Then, for any* $u \in U \cap \tilde{\omega}_1$ *and* $v \in V \cap \tilde{\omega}_2$, *there are* $u^* \in \mathrm{int}(U) \cap \tilde{\omega}_1$ *and* $v^* \in \mathrm{int}(V) \cap \tilde{\omega}_2$ *such that* $F_{uv} = F_{u^*v^*}$.

Proof. First we show that $F_{uv} = F_{u(0)v(0)}$ where $u(0) \in \tilde{\omega}_1 \cap U$ and $v(0) \in \tilde{\omega}_2 \cap V$ are suitable nonperiodic points. To do this take $u(0) = u$ if u is not periodic; otherwise by (ii) of Theorem 4.10 there is a nonperiodic point $u(0)$ in $U \cap \tilde{\omega}_1$ such that $\omega_f(u) = \omega_f(u(0))$, and one can easily verify that $u(0)$ can even be chosen such that $\liminf_{i\to\infty} \delta_{u(0)u}(i) = 0$. Then clearly $F_{uv} = F_{u(0)v}$. The point $v(0)$ is defined similarly.

Now let $m > 0$ be a common multiple of the periods of U and V. Since $f^m(U \cap \tilde{\omega}_1) = U \cap \tilde{\omega}_1$, there is a sequence $\{u(i)\}_{i=0}^{\infty}$ of points in $U \cap \tilde{\omega}_1$ such that $f^m(u(i+1)) = u(i)$ for any $i > 0$. Choose $\{v(i)\}_{i=0}^{\infty}$ in $V \cap \tilde{\omega}_2$ similarly. Now the point $u(i), v(i)$ are not periodic, hence for some j, $u(j) \in \mathrm{int}(U) \cap \tilde{\omega}_1$ and $v(j) \in \mathrm{int}(V) \cap \tilde{\omega}_2$. Take $u^* = u(j)$ and $v^* = v(j)$. □

6. Spectral decomposition

Also the results in this section are modifications of these found in [18].

Lemma 6.1. *Let* $\tilde{\omega}_1$, $\tilde{\omega}_2$ *be basic sets for* $f \in C(\mathbb{S},\mathbb{S})$. *Assume that there are periodic intervals* U, V *and countable set* $Q \subset \mathbb{S}^2$ *of pairs* (u,v) *such that*

$$f|_{\tilde{\omega}_1} \text{ is strongly transitive in } \mathrm{int}(U) \quad \text{and} \quad f|_{\tilde{\omega}_2} \text{ in } \mathrm{int}(V) \tag{5}$$

and furthermore, that

$$u \in \tilde{\omega}_1 \cap \mathrm{int}(U) \quad \text{and} \quad v \in \tilde{\omega}_2 \cap \mathrm{int}(V) \quad \text{if } (u,v) \in Q. \tag{6}$$

Then there are points $x \in \tilde{\omega}_1 \cap U$ *and* $y \in \tilde{\omega}_1 \cap V$ *such that, for any* $t > 0$,

$$F_{xy}(t) \leqslant \inf\bigl\{F_{uv}(t)\colon (u,v) \in Q\bigr\} \tag{7}$$

and

$$F^*_{xy}(t) \geqslant \sup\bigl\{F^*_{uv}(t)\colon (u,v) \in Q\bigr\}. \tag{8}$$

Proof. Let T be a countable set, dense in $[0,1]$, and such that, for any $(u,v)\in Q$ and any $t\in T$, both F_{uv} and F^*_{uv} are continuous at t. Let $\{t_j\}_{j=1}^{\infty}$ and $\{u(j),v(j)\}_{j=1}^{\infty}$ be sequences of points from T and Q, respectively, such that for any $t\in T$ and any $(u,v)\in Q$, $t=t_j$, $u=u(j)$ and $v=v(j)$ for infinitely many j.

Next, using induction, we define positive integers

$$k(1)<q(1)<k(2)<q(2)<\cdots<k(i)<q(i)<\cdots$$

and decreasing sequences $\{U_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ of compact intervals with

$$\lim_{i\to\infty}\operatorname{diam}(U_i)=\lim_{i\to\infty}\operatorname{diam}(V_i)=0,$$

and such that for any $u\in U_n$ and $v\in V_n$ and any $j\leqslant n$,

$$\frac{1}{k(j)}\xi\big(u,v,k(j),t_j\big)\leqslant F_{u(j)v(j)}(t_j)+\frac{1}{j} \tag{9}$$

and

$$\frac{1}{q(j)}\xi\big(u,v,q(j),t_j\big)\geqslant F^*_{u(j)v(j)}(t_j)-\frac{1}{j}. \tag{10}$$

To do this, we take $U_1=U$, $V_1=V$, $k(1)=1$, $q(1)=2$, and assume that U_n, V_n, $k(n)$ and $q(n)$ have been defined such that $f^j(U_n)\cap\omega_1$ and $f^j(V_n)\cap\omega_2$ are infinite whenever j is sufficiently large. Since U and V are periodic, by (5) and (6) there is some $s>q(n)$ such that $u(n+1)\in f^s(U_n)$ and $v(n+1)\in f^s(V_n)$. Let $a\in U_n$ and $b\in V_n$ be such that $f^s(a)=u(n+1)$ and $f^s(b)=v(n+1)$. Then clearly $F_{ab}=F_{u(n+1)v(n+1)}$ and $F^*_{ab}=F^*_{u(n+1)v(n+1)}$. Now the existence of $U_{n+1}\subset U_n$, $V_{n+1}\subset V_n$, $k(n+1)$ and $q(n+1)$ follows easily by Lemma 5.5. (We take as U_{n+1} and V_{n+1} compact neighborhoods of a and b, respectively, with $\operatorname{diam}(U_n)>2\operatorname{diam}(U_{n+1})$ and $\operatorname{diam}(V_n)>2\operatorname{diam}(V_{n+1})$. By (i) of Theorem 4.10, a, b, U_{n+1} and V_{n+1} can be chosen such that both $f^s(U_{n+1})\cap\tilde\omega_1$ and $f^s(V_{n+1})\cap\tilde\omega_2$ are infinite.)

Take $x'\in\bigcap_{j=1}^{\infty}U_j$ and $y'\in\bigcap_{j=1}^{\infty}V_j$. For any $t\in T$ and any $(u,v)\in Q$, take j such that $t=t_j$, $u=u(j)$ and $v=v(j)$. Since $x'\in U_j$ and $y'\in V_j$ (9) applies with $u=x'$ and $v=y'$. Since j can be arbitrarily large we have $F_{x'y'}(t)\leqslant F_{uv}(t)$. This implies (7) for $x=x'$, $y=y'$, any $t\in T$, and since T is dense in $[0,1]$, also for any t. The argument for (8) is similar.

Finally, let $w\in\tilde\omega_1\cap U$ be such that $\omega_f(w)=\tilde\omega_1$ (see (ii) of Theorem 4.10) and let $\{W_i\}_{i=1}^{\infty}$ be a decreasing sequence of compact neighborhoods of w with $\lim_{i\to\infty}W_i=w$. Since $f|_{\tilde\omega_1}$ is strongly transitive (cf. (vi) of Theorem 4.10) we can apply Lemma 4.2 and obtain $\omega_f(x')\subset\tilde\omega_1$. Similarly we get $\omega_f(y')\subset\tilde\omega_2$. Now by Lemma 5.2 we get $x\in\tilde\omega_1$ and $y\in\tilde\omega_2$ such that $\lim_{i\to\infty}\delta_{x'x}(i)=0$ and $\lim_{i\to\infty}\delta_{y'y}(i)=0$ which implies (7) and (8). □

Lemma 6.2 (cf. Lemma 5.4 in [18]). *Let $\{N_i\}_{i=0}^{\infty}$ be decomposition of the set $\mathbb{N}$ of positive integers into infinite sets. Then there is an uncountable Borel set $B\subset\{0,1\}^{\mathbb{N}}$ such that, for any distinct $\alpha=\{\alpha(i)\}_{i=0}^{\infty}$ and $\beta=\{\beta(i)\}_{i=0}^{\infty}$ in B and any n,*

$$\big\{j\in N_n\colon \alpha(j)\neq\beta(j)\big\}\quad\textit{is infinite.} \tag{11}$$

Lemma 6.3. *Let $f \in C(\mathbb{S}, \mathbb{S})$, and let $\tilde{\omega} = \omega_f(z)$ be a basic set. Let U be a minimal compact periodic interval with* $\mathrm{Orb}(U) \supset \tilde{\omega}$*, and let x_0, x_1 be in $U \cap \tilde{\omega}$. Then there is nonempty perfect set $P \subset \tilde{\omega}$ such that, for any distinct $u, v \in P$,*

$$F_{uv} \leqslant F_{x_0x_1} \quad \text{and} \quad F^*_{uv} \geqslant F^*_{x_0x_1}. \tag{12}$$

Proof. It is similar to that one for Lemma 5.5 in [18]. We use from Lemma 6.1, and methods of symbolic dynamics. Let T be a countable subset of $\mathbb{S}$, dense in $\mathbb{S}$ and such that both $F_{x_0x_1}$ and $F^*_{x_0x_1}$ are continuous at each $t \in T$, and let $\{t_j\}_{j=1}^{\infty}$ be a sequence of points from T that contains every t from T infinitely many times. Let $X_n = \{0,1\}^n$, for $n = 1, 2, \dots$, and define a system of compact intervals $\{I_\alpha\colon\ \alpha \in X_j\}_{j=1}^{\infty}$ and positive integers

$$k(1) < q(1) < k(2) < q(2) < \cdots < k(j) < q(j) < \cdots$$

such that, for every $\alpha = \alpha(1)\alpha(2)\cdots\alpha(n)$ and $\beta = \beta(1)\beta(2)\cdots\beta(n)$ in X_n, the following is true

$$f^j(I_\alpha) \cap \tilde{\omega} \quad \text{is infinite if} \quad j > k(n+1), \tag{13}$$

$$\text{if } \alpha \neq \beta \text{ then } I_\alpha \cap I_\beta = \emptyset, \tag{14}$$

$$\text{if } \gamma \text{ is in } X_k \text{ for some } k \quad \text{then } I_{\alpha\gamma} \subset I_\alpha \subset \mathrm{int}(U), \tag{15}$$

for any $u \in I_\alpha$ and $v \in I_\beta$, and any $j \leqslant n$,

$$\frac{1}{k(j)}\xi\bigl(u, v, k(j), t_j\bigr) \leqslant F_{x_{\alpha(j)}x_{\beta(j)}}(t_j) + \frac{1}{j} \tag{16}$$

and

$$\frac{1}{q(j)}\xi\bigl(u, v, q(j), t_j\bigr) \geqslant F^*_{x_{\alpha_j}x_{\beta_j}}(t_j) - \frac{1}{j}. \tag{17}$$

To do this, let I_0 and I_1 be disjoint compact subintervals of $\mathrm{int}(U)$, such that both $I_0 \cap \tilde{\omega}$ and $I_1 \cap \tilde{\omega}$ are infinite. Put $k(1) = 1$ and $q(1) = 2$ and assume by induction that $\{I_\alpha\colon\ \alpha \in X_n\}$, $k(n)$ and $q(n)$ have been defined. Assume that $f^j(I_\alpha) \cap \tilde{\omega}$ is infinite whenever $j > r$ and $\alpha \in X_n$. Let m be the period of U. By Lemma 5.6 we may assume that $x_0, x_1 \in \mathrm{int}(U)$ and since $f|_{\tilde{\omega}}$ is strongly transitive in $\mathrm{int}(U)$ ((vi) of Theorem 4.10), there is an $s > \max\{r, q(n)\}$ such that $x_0, x_1 \in \mathrm{int}(f^{mj}(I_\alpha))$ whenever $\alpha \in X_n$ and $j \geqslant s$. Since $\tilde{\omega}$ is perfect ((i) of Theorem 4.10), it is easy to see that, for $i = 0, 1$ and any $\alpha \in X_n$, there is a point $a(\alpha, i) \in \mathrm{int}(I_\alpha)$ such that $f^{ms}(a(\alpha, i)) = x_i$ and such that for any neighborhood V of $a(\alpha, i)$, $f^{ms}(V) \cap \tilde{\omega}$ is infinite.

Applying Lemma 5.5 we can find $q(n+1) > k(n+1) > ms$ and pairwise disjoint compact neighborhoods I_β of the points $a(\alpha, i)$ for all $\beta \in X_{n+1}$, where $\beta = \alpha i$ (we use αi for concatenation of α and i) such that (13)–(17) are satisfied when n is replaced by $n+1$.

Let $A = \bigcap_{n=1}^{\infty} \bigcup\{I_\alpha\colon\ \alpha \in X_n\}$. Define a map, code: $A \to X$, by $\mathrm{code}(x) = \alpha(1)\alpha(2)\cdots\alpha(n)\cdots$ if $x \in I_{\alpha(1)\alpha(2)\cdots\alpha(n)}$, for any n. Clearly code is a continuous map of A onto X. Moreover, code is constant on each connected component $J(\alpha) = \bigcap_{n=1}^{\infty} I_{\alpha(1)\alpha(2)\cdots\alpha(n)}$ of A; we have $\mathrm{code}(x) = \alpha$ for any $x \in J(\alpha)$. Thus if $A^* \subset A$ is a set that contains just one point from any connected component of A, then A^* is a Borel set and code is a continuous one-to-one map from A^* onto X. For $t \in T$, let $N_t = \{i \in \mathbb{N}\colon t_i = t\}$. Apply Lemma 6.2 to the decomposition $\{N_t\}_{t\in T}$ of $\mathbb{N}$; let B be corresponding set. Then $\mathrm{code}^{-1}(B) \cap A^*$ is an uncountable Borel set, hence it contains a nonempty perfect subset Q (cf., e.g., [10]).

Let $u, v \in Q$, $u \neq v$, let $\mathrm{code}(u) = \{\alpha(i)\}_{i=1}^{\infty}$ and let $\mathrm{code}(v) = \{\beta(i)\}_{i=1}^{\infty}$. By Lemma 6.2 there is an arbitrarily large j such that $\alpha(j) \neq \beta(j)$ and $t = t_j$. Hence (16) gives $F_{uv}(t) \leqslant F_{x_0x_1}(t)$, and since t is arbitrary in T, $F_{uv} \leqslant F_{x_0x_1}$. The argument for the second inequality in (12) is similar. Similarly, as at the end of proof of Lemma 6.1, we see that $\omega_f(u) \subset \tilde{\omega}$ for any $u \in Q$. By Lemma 5.2 there are $u^*, v^* \in \tilde{\omega}$ such that $\lim_{i\to\infty} \delta_{uu^*}(i) = 0$ and $\lim_{i\to\infty} \delta_{vv^*}(i) = 0$. Clearly Eq. (12) remains valid also for u^* and v^* instead of u and v. Since $\mathrm{code}(u^*) = \{\alpha^*(i)\}_{i=1}^{\infty}$ and $\mathrm{code}(v^*) = \{\beta^*(i)\}_{i=1}^{\infty}$ differ at infinitely many places Q and consequently, $\tilde{\omega}$ contains a nonempty perfect set with the required properties. □

Lemma 6.4. *Let $f \in C(\mathbb{S}, \mathbb{S})$ and let $\{\omega_i\}_{i=1}^{\infty}$ be the minimal periodic portions of basic sets of f. For any i, j, set $G_{ij} = \inf\{F_{uv}\colon u \in \omega_i,\ v \in \omega_j\}$. Then*

(i) *Each G_{ij} is zero on an interval $[0, \varepsilon(i, j)]$, where $\varepsilon(i, j)$ is a positive number.*
(ii) *The set $\{G_{ij}\colon \omega_i \cap \omega_j \neq \emptyset\}$ has a finite number of minimal elements.*
(iii) *The set $\{G_{ii}\}_{i=1}^{\infty}$ has a finite number of minimal elements.*

Proof. (i) By (iii) of Theorem 4.10, there are distinct periodic points p in ω_i and q in ω_j. Since $\min_s \delta_{pq}(s) = \varepsilon > 0$ we have $F_{pq}(t) = 0$, and $F_{pq} \geqslant G_{ij}$ implies $G_{ij}(t) = 0$ for $t \leqslant \varepsilon$. Take $\varepsilon(i, j) = \varepsilon$.

(ii) We may assume that $\omega_i \neq \omega_j$, for $i \neq j$. Denote $K_i = \mathrm{Env}(\omega_i)$, and $\varepsilon = \varepsilon(1, 1)$ (from assertion (i)). We say that an ω_i is *extremal*, if $\mathrm{diam}(\omega_i) > \varepsilon/2$ and if K_i is properly contained in no K_j. Note that there are only finitely many extremal ω_i's ((v) of Theorem 4.10). Let $\omega_1, \ldots, \omega_{n(1)}$ be all extremal ω_i. Note that $n(1) \geqslant 1$ since f has positive topological entropy (cf. [14]).

Let $m > 0$ be an integer. We say that an ω_i is *significant* if $\mathrm{diam}(\omega_i) > \varepsilon/2$ and the period of ω_i is less than m. By Lemma 5.3 there are only finitely many significant ω_i's. Without loss of generality we may assume that there are integers $n(3) \geqslant n(2) \geqslant n(1) > 0$ such that the system $\{\omega_1, \ldots, \omega_{n(2)}\}$ is invariant with respect to f, contains all extremal and all significant sets, and the system $\{\omega_{n(2)+1}, \ldots, \omega_{n(3)}\}$ consists of all ω_i such that it has a common point with some ω_j, for $j \leqslant n(2)$.

From definitions of significant and extremal ω_i it follows that $n(2)$ and $n(3)$ but not $n(1)$ depend on the parameter m. We show that the minimal elements of $\{G_{ij}\colon \omega_i \cap \omega_j \neq \emptyset\}$ are in the set $M = \{G_{ij}\colon i, j \leqslant n(3)\}$, for sufficiently large m. Let $i > n(3)$ and $\omega_i \cap \omega_j \neq \emptyset$. Then ω_j cannot be significant and consequently $j > n(2)$. Take $u \in \omega_i$ and $v \in \omega_j$ and show that $F_{uv} \geqslant G$, for some $G \in M$. If $\mathrm{diam}\, f^s(\omega_i \cup \omega_j) \leqslant \varepsilon$ for any s then $F_{uv}(t) = 1$ for $t > \varepsilon$, and hence $F_{uv} \geqslant G_{11}$.

So it suffices to consider the case $\operatorname{diam} f^s(\omega_i \cup \omega_j) > \varepsilon$, for some s. Since $\{\omega_1, \dots, \omega_{n(2)}\}$ is invariant with respect to f, and $i, j > n(2)$, we have $f^s(\omega_i), f^s(\omega_j) \notin \{\omega_1, \dots, \omega_{n(2)}\}$. Thus we may assume $\operatorname{diam}(\omega_i \cup \omega_j) > \varepsilon$, $i > n(3)$, $j > n(2)$, and

$$\operatorname{diam} f^s(\omega_i \cup \omega_j) \leqslant \operatorname{diam}(\omega_i \cup \omega_j) \quad \text{for any } s. \tag{18}$$

One of the sets ω_i, ω_j, say ω_i, has diameter $> \varepsilon/2$. Since $i > n(1)$, it is not extremal hence there is an extremal ω_r such that $\omega_i \subset \operatorname{int}(K_r)$ and consequently, $\omega_i \cup \omega_j \subset \operatorname{int}(K_r)$, since $\omega_i \cap \omega_j \neq \emptyset$ (see (v) of Theorem 4.10). By (i) and (iii) of Theorem 4.10 there are periodic points $a, b \in \omega_r$ (sufficiently close to end points of K_r) with the following property: If J is an interval such that

$$J \cap \omega_r \quad \text{is infinite}, \quad \operatorname{diam} J > \varepsilon, \text{ and } J \subset K_r \tag{19}$$

then $J \subset (a, b)$.

If $K_i \cap \omega_r$ would be infinite then the minimal compact periodical interval containing ω_i must contain K_r and, by Lemma 4.2 and (vi) of Theorem 4.10, $\omega_i = \omega_r$, which is impossible. Similarly $K_j \cap \omega_r$ is finite. Since $J = K_i \cup K_j$ satisfies (19) we have $\omega_i \cup \omega_j \subset (a, b)$. Let t_0 be such that $\operatorname{diam}(\omega_i \cup \omega_j) < t_0 < \varrho(a, b)$. Take $\lambda_r = F_{a,b}(t_0)$; clearly $\lambda_r < 1$. Let $\varepsilon(r, r)$ be as in (i). Then $G_{rr}(t) = 0$ for $t \leqslant \varepsilon(r, r)$ and, by (18), $G_{rr}(t) \leqslant 1 = F_{uv}$ for $t > t_0$. And if $\varepsilon(r, r) < t_0$ then by Lemma 5.4, $G_{rr}(t) \leqslant F_{ab}(t) \leqslant \lambda_r < F_{uv}(t)$ whenever $\varepsilon(r, r) < t < t_0$ and $m \geqslant n(\varepsilon(r, r), \lambda_r)$. Thus $G_{rr} \leqslant F_{uv}$ if $m \geqslant n(\varepsilon(r, r), \lambda_r)$.

When the parameter $m = \max\{n(\varepsilon(r, r), \lambda_r)\colon 1 \leqslant r \leqslant n(1)\}$ it follows that $\{G_{ij}\colon i, j \leqslant n(3)\}$ contains the minimal elements of $\{G_{ij}\colon \omega_i \cap \omega_j \neq \emptyset\}$.

(iii) It suffices to show that the minimal elements of $\{G_{ii}\}_{i=1}^{\infty}$ are contained in $\{G_{ij}\colon i \leqslant n(2)\}$. But it follows from the argument given above. $\square$

7. Proof of the main theorem

We give the proof of Theorem B. Theorem A is its particular case.

We follow the idea of the proof of Theorem 2.4 in [18].

(A) This result was already proved in [14] as Theorem 2.2.

(B) Let $\{\omega_i\}_{i=1}^{\infty}$ be system of the minimal periodic portions of all basic sets. (This system is nonempty since topological entropy is positive [14] and countable by (iv) of Theorem 4.10.) Denote by $\tilde{\omega}_u$ the maximal ω-limit set containing $\omega_f(u)$.

(B1) Let $D = \{F_{uv}\colon u \text{ and } v \text{ are synchronous}\}$ and $E = \{F_{uv}\colon u, v \in \omega_i, \text{ for some } i\}$. It is easy to see that $E \subset D$. To prove $D \subset E$ take $F_{uv} \in D$. If $\tilde{\omega}_u$ $(= \tilde{\omega}_v)$ is a solenoid then trajectories of u and v enter into periodical decomposition of arbitrarily high order and consequently $F_{uv} = \chi_{(0,\infty)} \in E$. For details see [14]. If $\tilde{\omega}_u$ is a basic set then by Lemma 5.2 there are $u^*, v^* \in \tilde{\omega}_u$ such that $F_{uv} = F_{u^*v^*}$. Thus $D = E$ and Lemma 6.4 gives the result on the spectrum $\Sigma(f)$.

Now let $D_w = \{F_{uv}\colon u, v \text{ satisfy } \liminf_{i\to\infty} \delta_{uv}(i) = 0\}$ and $E_w = \{F_{uv}\colon u \in \omega_i, v \in \omega_j \text{ and } \omega_i \cap \omega_j \neq \emptyset, \text{ for some } i \text{ and } j\}$. Let $F_{uv} \in D_w$. Similarly as before either $F_{uv} = \chi_{(0,\infty)}$ or there are points $u^* \in \tilde{\omega}_i$, $v^* \in \tilde{\omega}_j$ such that $\lim_{i\to\infty} \delta_{u^*u}(i) = 0$, $\lim_{i\to\infty} \delta_{v^*v}(i) = 0$ and this with

$$\liminf_{i\to\infty} \delta_{uv}(i) = 0 \tag{20}$$

gives $\tilde\omega_u \cap \tilde\omega_v \neq \emptyset$. Thus $F_{u,v} = F_{u^*v^*} \in E_w$. Consequently $D_w \subset E_w$.

To prove $E_w \subset D_w$, take $F_{uv} \in E_w$, $u \in \tilde\omega_i$, $v \in \tilde\omega_j$, $w \in \tilde\omega_i \cap \tilde\omega_j$ and $Q = \{(u,v),(w,w)\}$. Now apply Lemma 6.1 to get x, y such that $F_{xy} \leqslant F_{uv}$ and $F^*_{xy} = \chi_{(0,\infty)}$. Since $\liminf_{i\to\infty} \delta_{xy}(i) = 0$ we have $\omega_f(x) \cap \omega_f(y) \neq \emptyset$. Thus $F_{xy} \in D_w$. Since $D_w \subset E_w$ and E_w has the lower bounds in D_w, both D_w and E_w have the same system $\Sigma_w(f)$ of minimal elements and application of Lemma 6.4 on E_w completes the proof.

(B2) For any $k \leqslant m$ there are $x, y \in \omega_i$ for some i such that $F_{xy} = F_k$ (see proof (B1)). Existence of P_k now follows by Lemma 6.1 with $Q = \{(x,y),(x,x)\}$ and Lemmas 5.6 and 6.3.

(B4) Let any $u, v \in S$ satisfy (20). If for some $u \in S$, $\tilde\omega_u$ is solenoid then similarly as in the proof of (B1) we get $F_{uv} = \chi_{(0,\infty)}$ for any $u, v \in S$ and $S = S_0 \cup \emptyset \cup \emptyset$ is the corresponding decomposition.

Assume that for every $u \in S$ the set $\tilde\omega_u$ is a basic set. For every $u \in S$ denote by $u^* \in \tilde\omega_u$ a point such that $\lim_{n\to\infty} \delta_{u^*u}(n) = 0$ (Lemma 5.2). Let $T_i = \{u \in S\colon\ u^* \in \omega_i\}$, Lemma 5.2 shows that $S = \bigcup_{i=1}^{\infty} T_i$. Suppose that four distinct sets $T_{j(1)}$, $T_{j(2)}$, $T_{j(3)}$ and $T_{j(4)}$ are nonempty. Any two sets $\omega_{j(r)}$, $\omega_{j(s)}$, $1 \leqslant r, s \leqslant 3$, have a point in common (since $\liminf_{i\to\infty} \delta_{u(r)u(s)}(i) = 0$, for $u(i) \in T_i$). Now by (v) of Theorem 4.10 two of sets $\omega_{j(1)}, \omega_{j(2)}, \omega_{j(3)}$ and $\omega_{j(4)}$ must coincide (say $\omega_{j(3)}$ and $\omega_{j(4)}$).

By Lemma 6.4, for any $1 \leqslant k \leqslant 3$ and any $u, v \in T_{j(k)}$ there is $j \leqslant m$ such that $F_{uv} \geqslant G_{j(k)j(k)} \geqslant F_j$. Consequently $S = T_{j(1)} \cup T_{j(2)} \cup T_{j(3)}$.

(B3) If $j > m$ and $F_{uv} = F_j$ for any distinct $u, v \in S$ then u^* and v^* belong to different $T_{j(k)}$. Elsewhere F_{uv} cannot be minimal. This shows that every $T_{j(k)}$ contains one point. $\square$

References

[1] M. Babilonová, Distributional chaos for triangular maps, Ann. Math. Sil. 13 (1999) 33–38.

[2] A. Blokh, The "spectral decomposition" for one-dimensional maps, Dynamics reported, Dynam. Report. Expositions Dynam. Systems (N.S.) 4 (1995) 1–59.

[3] A. Blokh, On transitive maps of one-dimensional branched manifold, in: Differential-Difference Equations and Problems of Mathematical Physics, Kiev, 1984, pp. 3–9 (in Russian).

[4] A. Blokh, On dynamics on one-dimensional branched manifolds 1, Theory of Functions, Functional Analysis and Applications, Kharkov 46 (1986) 8–18 (in Russian).

[5] A. Blokh, On dynamics on one-dimensional branched manifolds 2, Theory of Functions, Functional Analysis and Applications, Kharkov 47 (1986) 67–77 (in Russian).

[6] A. Blokh, On dynamics on one-dimensional branched manifolds 3, Theory of Functions, Functional Analysis and Applications, Kharkov 47 (1987) 32–46 (in Russian).

[7] J. Bobok, Chaos in countable dynamics, Topology Appl., submitted for publication.

[8] J.S. Cánovas, Distributional chaos on tree maps: The star case, Comment. Math. Univ. Carolin. 42 (3) (2001) 583–590.

[9] G.-L. Forti, L. Paganoni, J. Smítal, Dynamics of homeomorphisms on minimal sets generated by triangular mappings, Bull. Austral. Math. Soc. 59 (1999) 1–20.

[10] C. Kuratowski, Topologie, Vol. I, PWN, Warsaw, 1958.

[11] S. Kolyada, L'. Snoha, Some aspects of topological transitivity—A survey, Grazer Math. Ber. 334 (1997) 3–35.

[12] M. Kuchta, Characterization of chaos for continuous maps of circle, Comment. Math. Univ. Carolin. 31 (1990) 388–391.

[13] Liao, Q. Fan, Minimal subshifts which display Schweizer–Smítal chaos and have zero topological entropy, Sci. China Ser. A 41 (1998) 33–38.
[14] M. Málek, Distributional chaos for continuous mappings of the circle, Ann. Math. Sil. 13 (1999) 205–210.
[15] A.N. Sharkovsky, The partially ordered system of attracting sets, Soviet Math. Dokl. 7 (1966) 1384–1386.
[16] A.N. Sharkovsky, The behavior of a map in neighborhood of an attracting set, Ukrajin. Mat. Ž. 18 (1966) 60–83 (in Russian).
[17] A.N. Sharkovsky, Continuous mappings on the set of ω-limit sets of iterated sequences, Ukrajin. Mat. Ž. 18 (1966) 127–130 (in Russian).
[18] B. Schweizer, J. Smítal, Measures of chaos and a spectral decomposition of dynamical systems on the interval, Trans. Amer. Math. Soc. 344 (1994) 737–754.
[19] B. Schweizer, A. Sklar, J. Smítal, Distributional (and other) chaos and its measurement, Real Anal. Exchange 27 (2001) 495–524.